Quantum and Classical Fidelity for Singular Perturbations of the Inverted and Harmonic Oscillator

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Abstract

Let us consider the quantum/versus classical dynamics for Hamiltonians of the form

$$H_g^\epsilon := \frac{P^2}{2} + \epsilon \frac{Q^2}{2} + \frac{g^2}{Q^2} \qquad (0.1)$$

where $\epsilon = \pm 1$, g is a real constant. We shall in particular study the Quantum Fidelity between H_g^ϵ and H_0^ϵ defined as

$$F_Q^\epsilon(t, g) := \langle \exp(-itH_0^\epsilon)\psi, exp(-itH_g^\epsilon)\psi \rangle \qquad (0.2)$$

for some reference state ψ in the domain of the relevant operators.

We shall also propose a definition of the Classical Fidelity, already present in the literature ([2], [3], [8], [12], [14]) and compare it with the behaviour of the Quantum Fidelity, as time evolves, and as the coupling constant g is varied.

1 INTRODUCTION

In the last few years, there has been a renewal of interest in the notion of "Quantum Fidelity" (also called Loschmidt Echo), in particular for applications in Quantum Chaos or Quantum Computation problems, (see for example ref. [11], [4], [9] and [7] for recent reviews including many earlier references). This notion is very simple because it amounts to considering the behaviour in time of the overlap of two quantum states: one evolved according to a given dynamics, the other one evolved by a slight perturbation of it, but starting from the **same initial state** at time zero. While this overlap obviously equals one at time zero, it starts decreasing as time evolves, although rather slightly if the size of the perturbation is small. Thus this phenomenon should rather be described as "sensitivity to perturbations" than by the naive denomination "fidelity", but we use it nevertheless since it has now become very common in the literature of the subject.

It is believed (and sometimes shown numerically) that the "generic" limiting value is zero, and that the type of decay (Gaussian or exponential) strongly depends on the chaotic versus regular classical underlying dynamics. However for some integrable systems strong recurrences to exact fidelity have been shown [13]. All these approaches are however rather heuristic, and these questions have not been treated rigorously, except recently by one of the authors [5].

In this paper we pursue a rigorous study of the Quantum Fidelity problem for rather simplistic Hamiltonian systems : Harmonic and Inverse Harmonic Oscillator perturbed by repulsive inverse quadratic potentials. The reference quantum states considered belong to a rather large class namely Perelomov's Generalized Coherent states of the $SU(1,1)$ algebra, suitable for the dynamics under consideration. This class, however large, is very specific, in particular the quantum (perturbed as well as unperturbed) dynamics can be exactly solved in terms of the classical trajectories of the **unperturbed** classical dynamics. Then the quantum fidelity can be shown to equal in absolute value the "return probability" for a very simple quantum dynamics and elementary reference states. Two very different behaviours in time are then demonstrated:
• periodic recurrences to 1 of the quantum fidelity for Harmonic Oscillator unperturbed dynamics
• exponential decrease in time to some asymptotic (non-zero) value in the future as well in the past for the Inverted Harmonic Oscillator unperturbed dynamics.

Then a similar notion already present in the physical literature ([2], [3], [8], [12], [14]) of "Classical Fidelity" is considered for these systems. It is just the overlap of two distribution functions in classical phase-space, one evolved by the unperturbed, the other one by the unperturbed classical dynamics. For the particular case of dynamics considered in this paper, we can evaluate the long time behaviour of the

Classical Fidelities for different distribution functions.

It is shown that a similar behaviour as for the quantum fidelities occurs, namely:

• periodic recurrences to 1 in the H. O. case

• fast decrease to some non-zero asymptotic value in the future as well in the past in the I. H. O. case.

However some differences, in particular in the neighborhood of initial time are demonstrated.

The plan of the paper is as follows:

In Section 2 we show that the quantum dynamics is exactly solvable in terms of the classical one for both the H. O. and I. H. O., and we introduce the class of quantum reference states under consideration. In section 3 (resp. 4),we describe the Quantum (resp. Classical) Fidelities for the H. O. case (stable case). In section 5 (resp. 6), we describe the Quantum (resp. Classical) Fidelities for the I. H. O. case (unstable case). In Section 7 we give concluding remarks. The Appendix contains the Proofs of the various estimates provided in Sections 4 and 6.

2 Quantum Fidelity for a suitable class of reference states (Perelomov Generalized Coherent States)

In all this section $\epsilon = \pm 1$. Furthermore the Hamiltonian H_g^{+1} (resp. H_g^{-1}) of equation (0.1) is simply denoted H_g^+ (resp. H_g^-)

According to [10], the evolution operator $\exp(-itH_g^\epsilon)\psi$ can be expressed explicitly in terms of solutions of the classical motion for H_0^ϵ, for any g. Here ψ belongs to a suitable class of wavepackets that we shall make precise below.

Let us denote by $z_\epsilon(t)$ the general form of a complex solution for Hamiltonian H_0^ϵ:

$$\ddot{z}_\epsilon(t) + \epsilon z_\epsilon(t) = 0 \qquad (2.1)$$

Its polar decomposition is written as:

$$z_\epsilon(t) := \exp(u_\epsilon + i\theta_\epsilon) \qquad (2.2)$$

with $t \mapsto u_\epsilon$, θ_ϵ being **real functions**. The constant wronskian of z_ϵ and $\bar{z}_\epsilon$ is taken as $2i$. This yields:

$$\dot{\theta}_\epsilon = e^{-2u_\epsilon} \qquad (2.3)$$

From equation (2.1), we easily deduce that $u_\epsilon(t)$ obeys the following differential equation:

$$\ddot{u}_\epsilon + \dot{u}_\epsilon^2 - e^{-4u_\epsilon} + \epsilon = 0 \qquad (2.4)$$

Let us denote by $D(u, v)$ the following unitary operator:

$$D(u, v) := \exp\left(\frac{-ivQ^2}{2}\right) \exp\left(\frac{iu(Q.P + P.Q)}{2}\right) \tag{2.5}$$

We shall now choose as reference wavepackets ψ the Generalized Coherent States for the $SU(1, 1)$ algebra (see [10]). This means that we consider the state ψ_0 which is the ground state of the operator H_g^+ namely

$$\psi_0(x) := c_g x^\alpha e^{-x^2/2} \tag{2.6}$$

c_g being a normalization constant such that $\|\psi_0\| = 1$ and α being determined by

$$\alpha := \frac{1}{2} + \sqrt{\frac{1}{4} + 2g^2} \tag{2.7}$$

one has

$$H_g^+\psi_0 = (\alpha + \frac{1}{2})\psi_0 \tag{2.8}$$

We know from [10] that ψ has the following general form

$$\psi_{u_0, v_0, \theta_0} := \exp\left(-i(\alpha + \frac{1}{2})\theta_0\right) D(u_0, v_0)\psi_0 \tag{2.9}$$

for general **real constants** θ_0, u_0, v_0.

Then we have proven the general result (see [5]):

Proposition 2.1 *Let u_ϵ, θ_ϵ be the functions defined above. Assume that they have the initial data*

$$\theta_\epsilon(0) = \theta_0, \ u_\epsilon(0) = u_0, \ \dot{u}_\epsilon(0) = v_0 \tag{2.10}$$

Then for any g we have:

$$e^{-itH_g^\epsilon}\psi = \psi_{u_\epsilon(t), \dot{u}_\epsilon(t), \theta_\epsilon(t)} \equiv \exp\left(-i\theta_\epsilon(t)(\alpha + \frac{1}{2})\right) D(u_\epsilon(t), \dot{u}_\epsilon(t))\psi_0 \tag{2.11}$$

which means that the set of Generalized Coherent States is stable under the quantum evolution generated by H_g^ϵ. Moreover

$$e^{-itH_0^\epsilon}\psi = e^{-i\theta_0(\alpha+\frac{1}{2})} D(u_\epsilon(t), \dot{u}_\epsilon(t)) \exp(-i(\theta_\epsilon(t) - \theta_0)H_0^+)\psi_0 \tag{2.12}$$

Remark 2.2 *The **same** functions u_ϵ, $v_\epsilon(t)$ and θ_ϵ appear in the formulas (2.11) and (2.12) above.*

We have the following important result:

Theorem 2.3 *For any real g and for $\epsilon = \pm 1$, we have:*

$$F_Q^\epsilon(t,g) = e^{-i\tilde{\theta}_\epsilon(t)(\alpha+1/2)} \langle \psi_0, \exp(i\tilde{\theta}_\epsilon(t)H_0^+)\psi_0 \rangle \qquad (2.13)$$

where by $\tilde{\theta}_\epsilon(t)$ we have denoted:

$$\tilde{\theta}_\epsilon(t) := \theta_\epsilon(t) - \theta_0 \qquad (2.14)$$

Proof: This follows easily from equation (2.11), and (2.12), and using the unitarity of the operator $D(u,v)$.
□

Thus the important fact to notice is that the modulus of the Quantum Fidelity (which is just the quantity refered to as Quantum Fidelity in the literature) reduces to the so-called **"return probability"** in the state ψ_0 for the reference states ψ_{u_0,v_0,θ_0} under consideration in this paper, for some rescaled time $\tilde{\theta}_\epsilon(t)$.

From now on, the modulus of the Quantum Fidelity functions for the case $\epsilon = +1$ (resp. $\epsilon = -1$) will be denoted as $F_Q(t,g)$ (resp. $G_Q(t,g)$). Thus:

$$F_Q(t,g) = |F_Q^{+1}(t,g)|, \quad G_Q(t,g) = |F_Q^{-1}(t,g)| \qquad (2.15)$$

3 Behavior of the Quantum Fidelity for $\epsilon = +1$

$H_0^+ = \frac{P^2+Q^2}{2}$ is simply the Harmonic Oscillator, whose classical solutions are linear combinations of $\cos t$ and $\sin t$.

The most general form of a complex solution is:

$$z(t) := (a+ib)\cos t + (c+id)\sin t \qquad (3.1)$$

with a, b, c, $d \in \mathbb{R}$. The constant wronskian of z and $\bar{z}$ is taken as $2i$. This yields:

$$ad - bc = 1 \qquad (3.2)$$

Writing $z_+ := e^{u_+ + i\theta_+}$, as in the previous section we get:

$$u_+(t) := \frac{1}{2}\log\{(a\cos t + c\sin t)^2 + (b\cos t + d\sin t)^2\} \qquad (3.3)$$

$$\tan\theta_+(t) := \frac{b\cos t + d\sin t}{a\cos t + c\sin t} \qquad (3.4)$$

Thus:

$$u_+(0) = \frac{1}{2}\log(a^2+b^2), \quad \dot{u}_+(0) = \frac{ac+bd}{a^2+b^2}, \quad \theta_+(0) = \arctan(\frac{b}{a}) \qquad (3.5)$$

Clearly we have the following general result for $F_Q^{+1}(t,g)$:

Proposition 3.1

$$F_Q^{+1}(t, g) = e^{-i\alpha\tilde{\theta}_+(t)} \sum_{n=0}^{\infty} |\lambda_n|^2 e^{in\tilde{\theta}_+(t)} \tag{3.6}$$

where λ_n are the coefficients of the expansion of ψ_0 in the eigenstates ϕ_n of H_0^+, and of course

$$\sum_{n=0}^{\infty} |\lambda_n|^2 = 1 \tag{3.7}$$

This expansion is finite and involves only even terms $\exp(2in\tilde{\theta}_+(t))$ in the particular case where g is of the form $g = \sqrt{k(k+1)/2}$ for $k = 1, 2, ...$

The proof is an immediate consequence of equation (2.13).□

Recall that

$$\phi_n(x) = (\sqrt{\pi}n!2^n)^{-1/2} e^{-x^2/2} H_n(x) \tag{3.8}$$

where H_n are the Hermite polynomials $H_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2}$

Let us consider the following cases $g = 1, \sqrt{3}, \sqrt{10}$ (which yields $\alpha = 1, 2, 3$ respectively). Then we have

$$F_Q^{+1}(t, 1) = \frac{2}{3} + \frac{1}{3} \exp(2i\tilde{\theta}_+(t)) \qquad F_Q^{+1}(t, \sqrt{3}) = \frac{2}{5} + \frac{3}{5} \exp(2i\tilde{\theta}_+(t)) \tag{3.9}$$

$$F_Q^{+1}(t, \sqrt{10}) = \frac{8}{35} + \frac{24}{35} e^{2i\tilde{\theta}_+(t)} + \frac{3}{35} e^{4i\tilde{\theta}_+(t)} \tag{3.10}$$

Study of $\tilde{\theta}_+(t)$

Of course $F_Q(t, g)$ depends on the reference state ψ via the coefficients λ_n. Clearly $\tilde{\theta}_+(t)$ is **independent of** g and we have the following property:

Lemma 3.2 $\tilde{\theta}_+(2\pi) = 2\pi$, and thus $F_Q(t, g)$ is 2π-periodic in t for any $g \in \mathbb{R}$. Moreover for $g = 1, \sqrt{3}, \sqrt{10}$,

$$F_Q(k\pi, g) = 1, \qquad \forall k \in \mathbb{Z} \tag{3.11}$$

Proof:

$$\int_0^\pi ds \, e^{-2u_+(s)} = \int_{-\pi/2}^{+\pi/2} ds \, e^{-2u_+(s)} = \int_{-\infty}^{+\infty} \frac{dx}{(c^2 + d^2)(x + \frac{ac+bd}{c^2+d^2})^2 + \frac{1}{c^2+d^2}} = \pi \tag{3.12}$$

□

$\tilde{\theta}(k\pi) = 0$, (mod π) which implies that for $g = 1, \sqrt{3}, \sqrt{10}...$, $F_Q(t, g)$ is π-periodic.

The minimum (in absolute value) is attained when $\tilde{\theta} = \frac{\pi}{2}$, ie for those values of t such that

$$\tan \theta(t) = \frac{b + d \tan t}{a + c \tan t} = -\frac{1}{\tan \theta(0)} \tag{3.13}$$

which holds if and only if $t = -\arctan\left(\frac{a^2+b^2}{ac+bd}\right) = -\arctan\left(\frac{1}{\dot{u}(0)}\right)$ We thus have

$$\min F_Q(t,1) = \frac{1}{3}, \quad \min F_Q(t,\sqrt{3}) = \frac{1}{5}, \quad \min F_Q(t,\sqrt{10}) = \frac{13}{35} \tag{3.14}$$

We now present the picture of the modulus of the Quantum Fidelity for $g = 1$, and constants a, b, c, d chosen as $a = d = -c = 1$, $b = 0$:
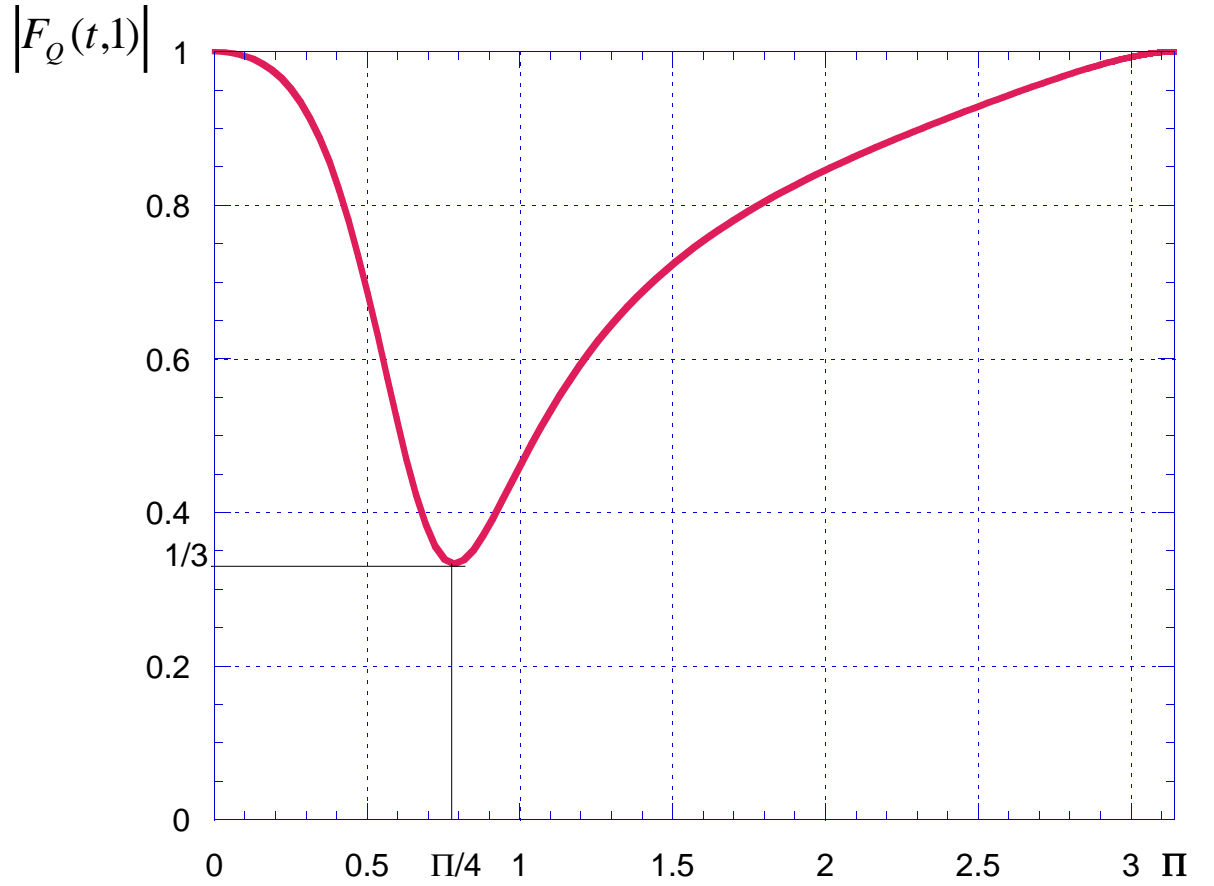

Fig. 1 : Quantum Fidelity (Stable Case)

A more symmetrical picture is obtained by taking $a = d = 1$, $b = c = 0$, in which case $\tilde{\theta}_+(t) \equiv t$, and the minimum of $F_Q(t,g)$ for $g = 1$, $\sqrt{3}$, $\sqrt{10}$.... is attained for $t = \pi/2$.

4 Comparison with the Classical Fidelity for H_g^+

In the literature various definitions of the Classical Fidelity have been proposed and studied, (see [2], [3], [8], [12], [14]). Let us here introduce the notion and make a choice of Definition, for which some exact estimates can be performed, together with numerical computations, in order to compare its behavior in time with the corresponding Quantum Fidelity.

We first show that the classical trajectories for H_g^+ can be simply deduced from those for H_0^+, and that a natural scaling makes them independent from the constant g. Let us make the scaling $(q, p) \mapsto (g\sqrt{2})^{1/2}(q, p)$.

Proposition 4.1 *Consider the complex solution (3.1) of $\ddot{z} + z = 0$ with wronskian of $z, \bar{z}$ equal to $2i$. Take $a = q \neq 0$, $b = 0$, $c = p$, $d = \frac{1}{q}$. Then we have:*

$$y(t) := \left((q\cos t + p\sin t)^2 + \frac{\sin^2 t}{q^2} \right)^{1/2} \tag{4.1}$$

is a solution of differential equation $\ddot{y} + y - \frac{1}{y^3} = 0$ and it has the same initial data as $x(t) := q\cos t + p\sin t$, namely $y(0) = x(0)$, $\dot{y}(0) = \dot{x}(0)$

Proof: An easy computation, using that $\dot{\theta}(t) = e^{-2u}$ shows that:

$$\ddot{z} = \left[\dot{u}^2 + \ddot{u} - e^{-4u} + i(\ddot{\theta} + 2\dot{u}\dot{\theta}) \right] = (\dot{u}^2 + \ddot{u} - e^{-4u})z \tag{4.2}$$

and thus $\ddot{z} + z = 0$ implies $e^u(1 + \dot{u}^2 + \ddot{u}) - e^{-3u} = 0$, whence the result, using that $y = e^u$ is such that $\ddot{y} = (\ddot{u} + \dot{u}^2)y$. Furthermore it is easy to check that $y = e^u$ is nothing but (4.1).
$\square$

Remark 4.2 *Reintroducing the scaling: $y' = (g\sqrt{2})^{1/2}y$, we check that $y'(t) := \left((q\cos t + p\sin t)^2 + \frac{2g^2 \sin^2 t}{q^2} \right)^{1/2}$ is a trajectory for the Hamiltonian H_g^+, namely obeys $\ddot{y}' + y' - \frac{2g^2}{y'^3} = 0$. Thus $(y'(t), \dot{y}'(t))$ is the classical phase space point of classical trajectory for H_g^+ that emerges from the same initial point (q, p) as $(q(t), p(t))$, (by continuity we set $y'(0) = 0$ if $q = 0$).*

We now define the Classical Fidelity as:

$$F_C(t, g) := \int_{\mathbb{R}^2} \rho(p(t), q(t)) \, \rho(\dot{y}'(t), y'(t)) \, dqdp \tag{4.3}$$

where ρ are suitably defined distribution functions in classical phase, satisfying

$$\int_{\mathbb{R}^2} dp \; dp \; \rho^2(p, q) = 1 \tag{4.4}$$

We shall make two different choices:

1) $\rho(p, q) = G(p, q) \equiv \frac{1}{\sqrt{\pi}} \exp\left(-\frac{p^2 + q^2}{2}\right)$ in which case the Classical Fidelity is denoted $F_C(t, g)$

2) $\rho(p, q) = X(p, q) \equiv \frac{1}{\sqrt{\pi}} \chi(p^2 + q^2 \leq 1)$ in which case the Classical fidelity is denoted $\tilde{F}_C(t, g)$, where χ is the characteristic function of the set indicated (here a disk of radius 1).

Clearly (due to the parity of ρ), $F_C(t, g)$ and $\tilde{F}_C(t, g)$ are π- periodic and we have

$$F_C(0, g) = \tilde{F}_C(0, g) = 1, \; \forall g \in \mathbb{R} \text{ and } \tilde{F}_C(t, 0) = F_C(t, 0) \equiv 1, \; \forall t \tag{4.5}$$

Since the Energy is conserved, we have $p(t)^2 + q(t)^2 = p^2 + q^2$ and

$$y'(t)^2 + \dot{y}'(t)^2 = p^2 + q^2 - \frac{2g^2}{y'(t)^2} + \frac{2g^2}{q^2} \tag{4.6}$$

so that:

$$F_C(t) = \frac{1}{\pi} \int_{\mathbb{R}^2} dq \; dp \; \exp\left(-q^2 - p^2 + \frac{g^2}{(q \cos t + p \sin t)^2 + \frac{2g^2}{q^2} \sin^2 t} - \frac{g^2}{q^2}\right) \tag{4.7}$$

$$\tilde{F}_C(t) = \frac{1}{\pi} \int_{p^2 + q^2 \leq 1} dq \; dp \; \chi(y^2(t) + \dot{y}(t)^2 \leq 1) \tag{4.8}$$

$F_C(t)$ is minimum for $t = \frac{\pi}{2}$ and its minimum equals

$$F_C(\frac{\pi}{2}) = \frac{1}{\pi} \int_{\mathbb{R}^2} dq \; dp \; \exp\left(-q^2 - p^2 + \frac{g^2}{(p^2 + \frac{2g^2}{q^2})} - \frac{g^2}{q^2}\right) \tag{4.9}$$

We shall now perform estimates of $F_C(t)$ and $\tilde{F}_C(t, g)$ and a fine analysis of the behaviour of $F_C(t, g)$ in the neighborhood of $t = 0$.

Proposition 4.3 (i) $F_C(t, g)$ is π-periodic and we have the following estimates:

$$e^{-2g} \leq F_C(t) \leq \inf\left(1, \sqrt{2} \exp\left(\frac{-g|\sin t|}{\sqrt{1 + \sin^2 t}}\right)\right) \tag{4.10}$$

(ii) $\tilde{F}_C(t, g)$ is π-periodic and we have the following uniform lower bound:

$$\tilde{F}_C(t, g) \geq 1 - 2g\sqrt{2} \tag{4.11}$$

(iii) If $g \geq 1/\sqrt{2}$, then $\tilde{F}_C(t, g)$ attains at $t = \pi/2$ its minimum which equals 0.

$$g \geq \frac{1}{\sqrt{2}} \implies \tilde{F}_C(\frac{\pi}{2}, g) = 0 \tag{4.12}$$

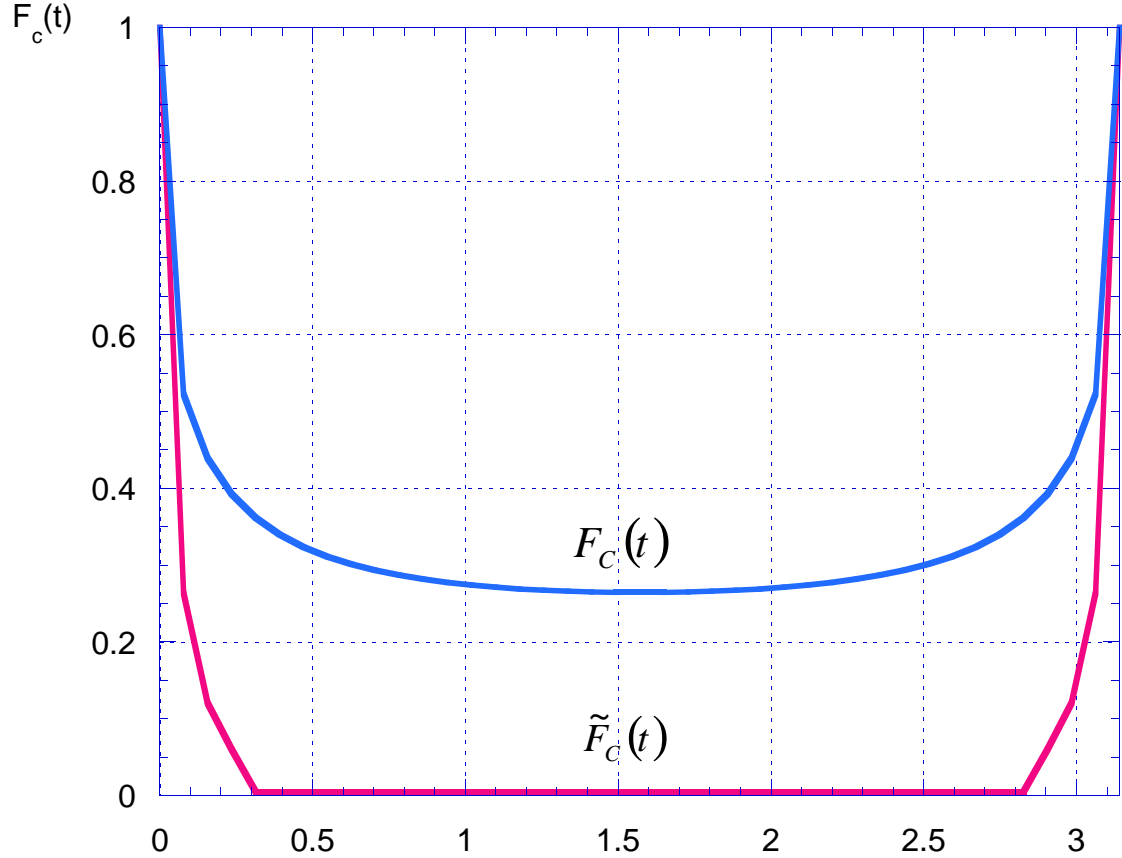The Proof of Proposition 4.3 is postponed to the Appendix.

Fig. 2 : Classical Fidelity (Stable Case, g=1)

Remark 4.4 *The lower bound in Proposition 4.3 is uniform in t. It shows, as expected, that the smaller g is, the closer to 1 is the Classical Fidelity. The upper bound shows that when g is large, then the Classical Fidelity can attain very small values in the interval between the recurrence values $t = k\pi$, $k \in \mathbb{Z}$. Thus although being π-periodic in time, and thus **strongly recurrent**, the Classical Fidelity can become very small for some values of t when g is large enough.*

We now consider the "generic" Classical Fidelity, independent of g which appears naturally under the scaling $p \to (g\sqrt{2})^{1/2}p$, $q \to (g\sqrt{2})^{1/2}q$. Then we consider a "rescaled" distribution function:

$$\rho(p, q) = G(p, q) \equiv \frac{1}{(\pi g\sqrt{2})^{1/2}} \exp\left(-\frac{p^2 + q^2}{2g\sqrt{2}}\right) \tag{4.13}$$

Under this scaling, we get a g-independent fidelity which is actually

$$F_C(t) \equiv F_C(t, \frac{1}{\sqrt{2}}) \tag{4.14}$$

We now give the precise behavior of $F_C(t)$ as $t \simeq 0$:

Proposition 4.5 *Let $A > 0$ any positive constant. Then, as $t \to 0$, we have:*

$$F_C(t) \sim e^{-A|t|}$$

The Proof of Proposition 4.5 is postponed to the Appendix.

Remark 4.6 *This estimate of $F_C(t)$ in the neighborhood of $t = 0$ shows a sharp decay, faster than exponential from the initial value $F_C(0) = 1$ (and due to the π-periodicity in time, this will reproduce at $t = k\pi, \quad \forall k \in \mathbb{Z}$). Thus the Classical Fidelity has cusps at those values of t. Proposition (4.5) (ii) shows that for g small enough, then the Classical Stability function is very close to 1, as expected. Proposition (4.5) (iii) shows that for g large enough, then this Classical Fidelity function vanishes in the interval $t \in [0, \pi]$, which expresses that at this point the Classical Fidelity is very bad.*

Let us now present the curves $F_C(t)$ and $\tilde{F}_C(t)$ on the same diagram, for $t \in [0, \pi]$; (these curves are obviously π-periodic).
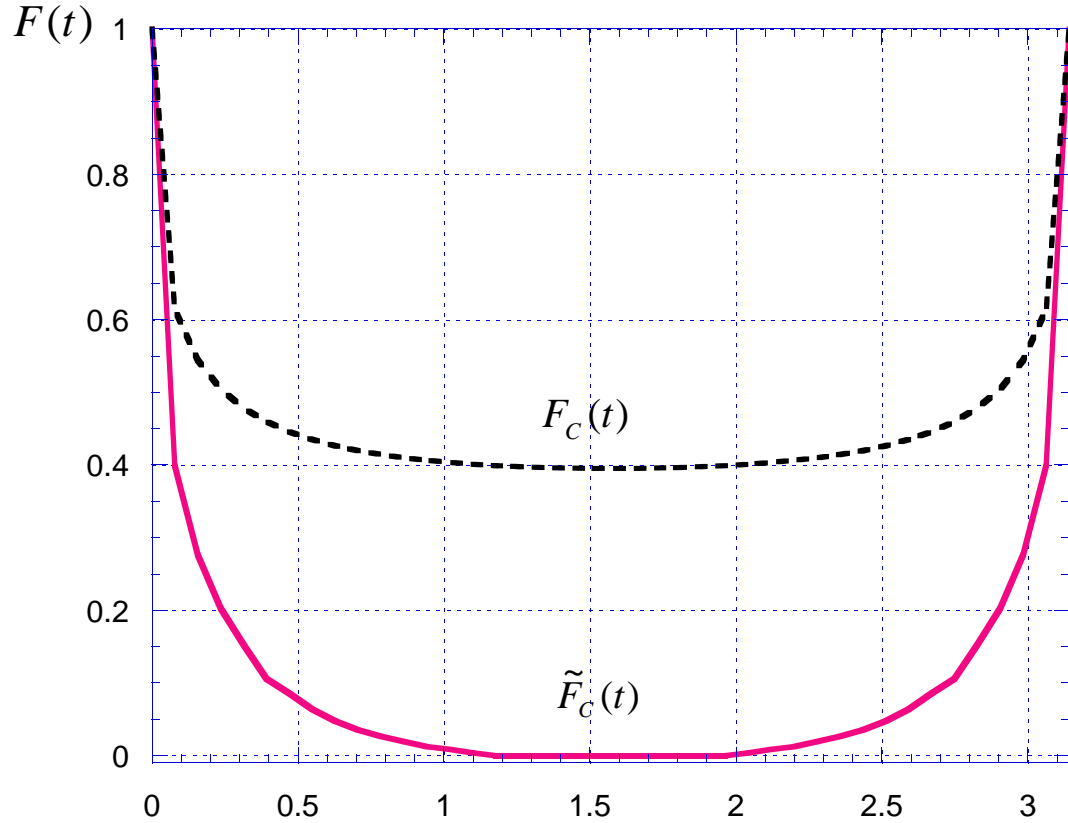
Fig. 3 : Classical Fidelity Functions (Stable Case)

5 The Quantum Fidelity for $\epsilon = -1$

The most general form of solution $z_{-1}(t)$ of $\ddot{z} - z = 0$ is

$$z_{-1}(t) = (a + ib) \cosh t + (c + id) \sinh t \tag{5.1}$$

a, b, c, d being real constants. Again we assume that the Wronskian of z and $\bar{z}$ equals $2i$, which yields (3.2).
Let $z_{-1} := e^{u_{-1} + i\theta_{-1}}$ be the polar decomposition of z_{-1}. Thus $u_{-1}(t)$ and $\theta_{-1}(t)$ are **real** functions of t, and satisfy:

$$\dot{\theta}_{-1}(t) = e^{-2u_{-1}(t)} \tag{5.2}$$

Here $\theta_{-1}(t)$ is such that

$$\tan \theta_{-1}(t) = \frac{b \cosh t + d \sinh t}{a \cosh t + c \sinh t} \tag{5.3}$$

Here the important fact is that the quantum propagators for the Inverted Harmonic (perturbed or unperturbed) Oscillator are given in terms of the Quantum Propagators for the NON-INVERTED Harmonic Oscillator (perturbed or unperturbed). The time functions that enter this decomposition are precisely the functions u_{-1}, θ_{-1} defined above (formula (5.3)). It was established in Section 2, together with the resulting Quantum Fidelity for a conveniently chosen set of reference states ψ. We have:

$$G_Q(t, g) \equiv |F_Q^{-1}(t, g)| = |\langle \psi_0, e^{i\tilde{\theta}_{-1}(t)H_0^+} \psi_0 \rangle| \tag{5.4}$$

Thus the time dependance of $F_Q^{-1}(t, g)$ is governed by the function $\tilde{\theta}_{-1}(t) \equiv \theta_{-1}(t) - \theta_{-1}(0)$ which is independent of g.

Study of $\tilde{\theta}_{-1}(t)$

Let us choose $b = 0$ for simplicity, so that $\theta_{-1}(0) = 0$ and thus $\tilde{\theta}_{-1}(t) = \theta_{-1}(t)$ in this case. Then

$$\tan \theta_{-1}(t) = \frac{\sinh t}{a(a \cosh t + c \sinh t)} \tag{5.5}$$

which converges exponentially fast to $\frac{\pm 1}{a(a \pm c)}$ as $t \to \pm\infty$.
This implies that the Quantum Fidelity never recurs to 1 in this case but tends exponentially fast to some constants at $\pm\infty$.

$$\cos(2\theta_{-1}(t)) \to \frac{a^2(a \pm c)^2 - 1}{a^2(a \pm c)^2 + 1} \tag{5.6}$$

as $t \to \pm\infty$, exponentially fast, so that in the case $g = 1$, we get a very simple form of $G_Q(t, 1)$. Let us draw the function $t \mapsto G_Q(t, 1)$, in the particular case $a = -c = 1$:
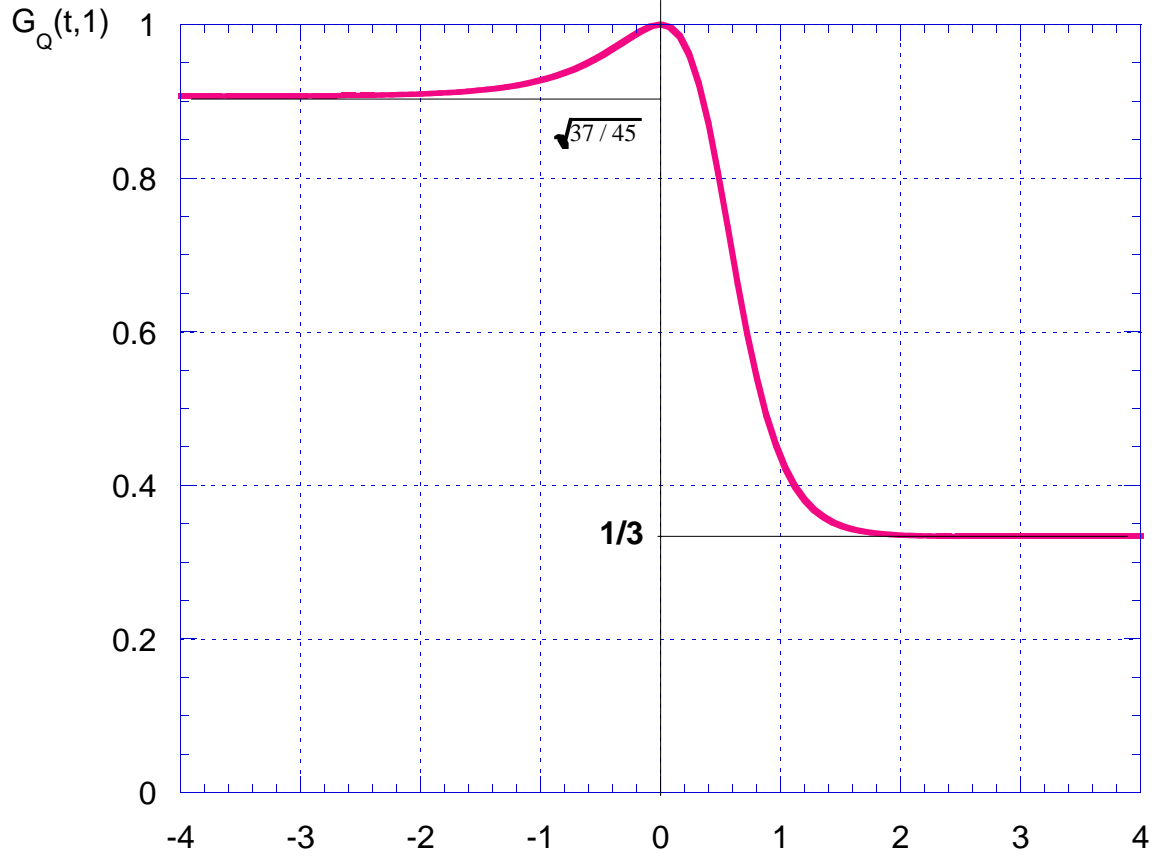

Fig. 4 : Quantum Fidelity (Inverted Osccilator)

In the case $a = d = 1$, $b = c = 0$, we get

$$\tan \theta_{-1}(t) = \tanh t \tag{5.7}$$

so that we have a more symmetrical behavior between the future and the past of the Quantum Fidelity:

$$G_Q(t, 1)^2 = \frac{5}{9} + \frac{4}{9 \cosh 2t} \sim \frac{5}{9} + \frac{8}{9} e^{-2|t|} \tag{5.8}$$

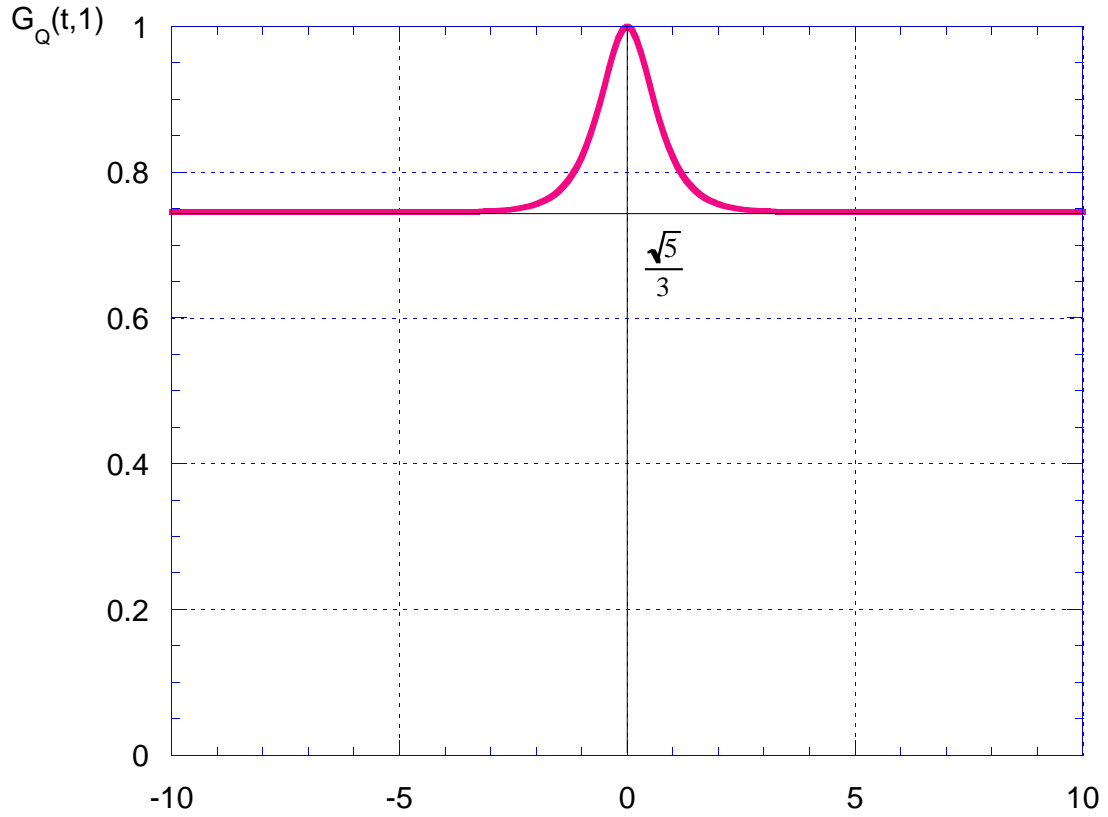The graph of $G_Q(t, 1)$ is then as follows:

Fig. 5 : Quantum Fidelity (Unstable Symmetric Case)

Remark 5.1 *The Quantum Fidelity, which we have denoted $G_Q(t,1)$ in this case is always bounded from below by 1/3, uniformly in t. A similar result holds true for $G_Q(t, \sqrt{3})$, and for $G_Q(t, \sqrt{10})$.*

6 Comparison with the Classical Fidelity for H_g^-

Here again we have that the classical trajectories for H_g^- can easily be deduced from that for H_0^-, and that the natural scaling $y(t) = (g\sqrt{2})^{1/2}y(t)$ makes them independent from g.

Proposition 6.1 *Let $z_{-1}(t)$ be given by (5.1) with Wronskian of $(z_{-1}, \ \bar{z}_{-1})$ being equal to $2i$ which yields (3.2). If $z_{-1} = e^{u_{-1}+i\theta_{-1}}$ is the polar decomposition of z, with $\theta_{-1}(0) = 0$, then , given initial data q, p with $q \neq 0$,*

$$z_{-1}(t) = q\cosh t + p\sinh t + i\frac{\sinh t}{q} \tag{6.9}$$

is such that $\Re z_{-1}(t) := x(t)$ obeys the differential equation $\ddot{x} - x = 0$, and

$$x(0) = q, \ \dot{x}(t) = p \tag{6.10}$$

and

$$y(t) := |z(t)| \tag{6.11}$$

is a trajectory for $\frac{p^2}{2} - \frac{q^2}{2} + \frac{1}{q^2}$, namely obeys

$$\ddot{y} - y - \frac{1}{y^3} = 0 \tag{6.12}$$

with the same initial data as $x(t)$:

$$y(0) = q, \ \dot{y}(0) = p \tag{6.13}$$

Remark 6.2 *We recover the general trajectory for H_g^- by simply rescaling the initial data $(q, \ p)$ by a factor $(g\sqrt{2})^{-1/2}$, which provides solutions of*

$$\ddot{y}' - y' - \frac{2g^2}{y'^3} = 0 \tag{6.14}$$

of the form:

$$y'(t) = \left((q\cosh t + p\sinh t)^2 + \frac{2g^2 \sinh^2 t}{q^2} \right)^{1/2} \tag{6.15}$$

Proof of Proposition 6.1:
It is essentially equivalent to that of Proposition 4.1.

We now define the Classical Fidelity in terms of distribution functions $\rho(p,q)$ as in Section 2 for the stable case. We denote them by $G_C(t,g)$ and $\tilde{G}_C(t,g)$ for $\rho = G(p,q)$, $X(p,q)$ respectively, (namely the Gaussian and characteristic functions of phase-space variables (q,p) that we have introduced in Section 4). However here, we assume that

$$X(p,q) := \frac{1}{\sqrt{3\pi}} \, \chi(p^2 + q^2 \leq 3\pi) \tag{6.16}$$

Proposition 6.3 *We have the following uniform in t lower bounds for $G_C(t,g)$, $\tilde{G}_C(t,g)$:*

$$G_C(t,g) \geq e^{-2g} \tag{6.17}$$

$$\tilde{G}_C(t,g) \geq 1 - \frac{2g\sqrt{2}}{3} \tag{6.18}$$

The Proof is postponed to the Appendix.

Remark 6.4 *Again for g small the Classical Fidelities $G_C(t,g)$, $\tilde{G}_C(t,g)$ remain close to 1 uniformly in time. We now explain why we have chosen the characteristic function of the ball $\{(p,q) \in \mathbb{R}^2 : \ p^2 + q^2 \leq 3\}$ instead of the ball of radius 1. We have generic **g-independent** Classical Fidelities that we denote $G_C(t)$, $\tilde{G}_C(t)$ by using the rescaled variables $p(g\sqrt{2})^{1/2}$, $q(g\sqrt{2})^{1/2}$. Under the rescaling of the Gaussian functions and characteristic functions used in the definitions, we find simple estimates as seen below.*

Proposition 6.5 *Let $x(t)$, $y(t)$ be as in Proposition 6.1. Then define:*

$$G_C(t) := \frac{1}{\pi} \int dq\, dp\, \exp\left(-\frac{x^2(t) + \dot{x}(t)^2 + y^2(t) + \dot{y}(t)^2}{2}\right) \tag{6.19}$$

$$\tilde{G}_C(t) := \frac{1}{3\pi} \int dp\, dq\, \chi(x^2(t) + \dot{x}(t)^2 \leq 3)\, \chi(y(t)^2 + \dot{y}(t)^2 \leq 3) \tag{6.20}$$

(i) We have the uniform lower bounds:

$$G_C(t) \geq e^{-\sqrt{2}} \tag{6.21}$$

$$\tilde{G}_C(t) \geq \frac{1}{3} \tag{6.22}$$

(ii) As $t \to \infty$, the limiting value of $\tilde{G}_C(t)$ satisfies the following estimate:

$$\tilde{G}_C(\infty) \leq \frac{2\arctan(\sqrt{17})}{\pi} - \frac{\sqrt{17}}{9\pi} \simeq 0.497 \tag{6.23}$$

The Proof is postponed to the Appendix.

This shows that the Classical Fidelity in this case doesn't decay to zero but instead stays bounded below by some constant. We now present on the graphic below the graph of $G_C(t)$:
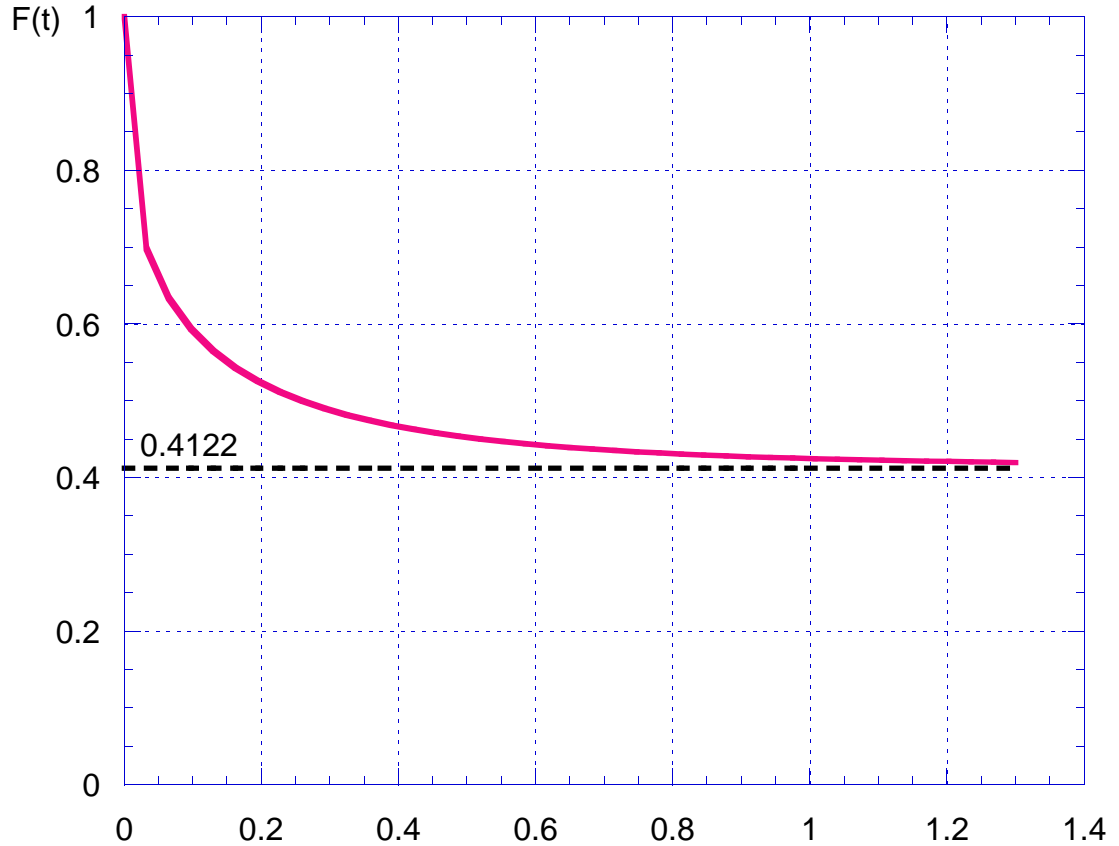
Fig. 6 : Classical Fidelity (Unstable Case)

Owing to equations (6.19) and (6.20), the Classical Fidelities have asymptotic values at $t \to \infty$ (thus $\tau := (\cosh 2t)^{-1} = 0$) equal respectively to

$$G_C(\infty) = \frac{1}{\pi} \int_{\mathbb{R}^2} du\ dv\ \exp\left(-u^2 - v^2 - \frac{1}{2u^2} + \frac{1}{(u+v)^2 + \frac{1}{u^2}}\right) \simeq 0.414 \quad (6.24)$$

$$\tilde{G}_C(\infty) = \frac{1}{3\pi} \int_{\mathbb{R}^2} du\ dv\ \chi(u^2+v^2 \le 3)\ \chi\left(u^2 + v^2 + \frac{1}{u^2} - \frac{2}{(u+v)^2 + \frac{1}{u^2}} \le 3\right) \simeq 0.497$$
$$(6.25)$$

Furthermore let us give simple expressions of the g-dependent Classical Fidelities, using the scaling noted above and the g-independent functions $y(t)$ given in Proposition 6.1. It appears that these expressions are more suitable for the numerical investigation of the dependence in parameter g of $G_C(t,g)$ and $\tilde{G}_C(t,g)$:

$$\frac{g\sqrt{2}}{\pi} \int_{\mathbb{R}^2} du\ dv \exp\left(-g\sqrt{2}\left[u^2 + v^2 + \frac{1}{u^2} - \frac{2}{u^2(1+\tau) + v^2(1-\tau) + (1-\tau)\frac{1}{u^2} + 2uv\sqrt{1-\tau^2}}\right]\right)$$
$$(6.26)$$
$$= G_C(t,g)$$

$$\tilde{G}_C(t,g) = \frac{g\sqrt{2}}{3\pi} \int_{\mathbb{R}^2} du\ dv\ \chi(u^2 + v^2 \le \frac{3}{g\sqrt{2}}) \quad (6.27)$$

$$\times \chi\left(u^2 + v^2 + \frac{1}{2u^2} - \frac{1}{u^2(1+\tau) + v^2(1-\tau) + (1-\tau)\frac{1}{u^2} + 2uv\sqrt{1-\tau^2}} \le \frac{3}{g\sqrt{2}}\right)$$

Recall that

$$\tau := \frac{1}{\cosh 2t}$$

We now draw on the same picture the Classical Fidelity functions $G_C(t)$ and $\tilde{G}_C(t)$: (we have not been able to prove rigorously that they actually monotonically decay to these asymptotic value as t goes to ∞). Next we draw on the same graphic the Classical and Quantal Fidelities for the specific value $g = 1$.
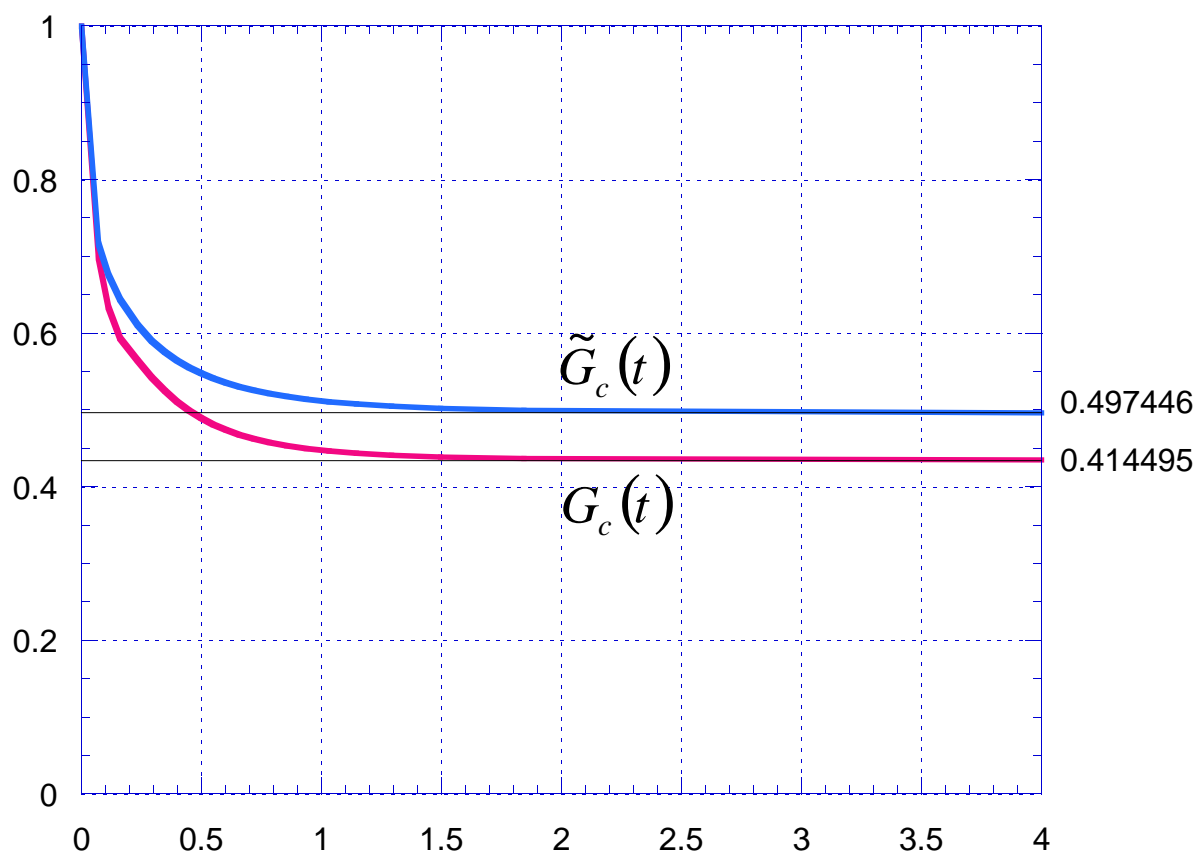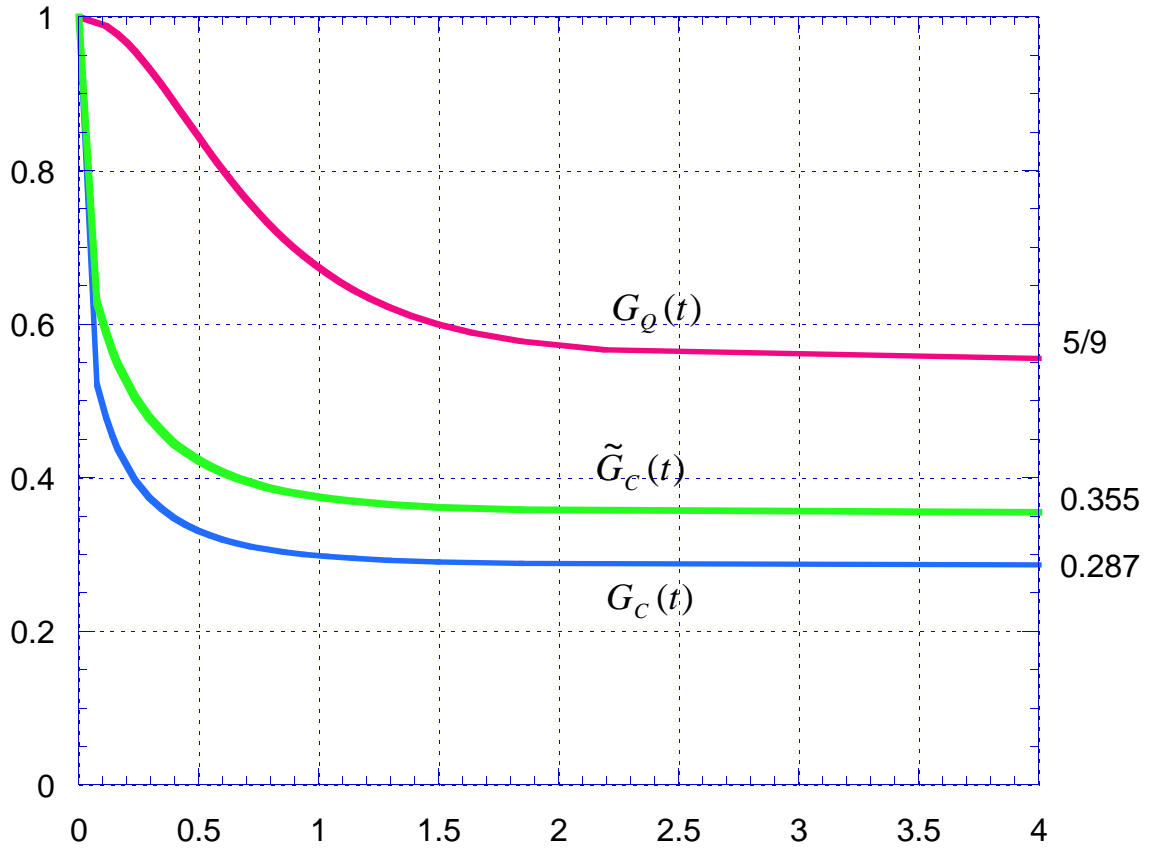
Fig. 7 : Classical Fidelity (Unstable Case g=1)

Fig. 8 : Quantum Classical fidelity Comparison (Unstable case g=1)

7 CONCLUDING REMARKS

In this paper we show the behaviour in time of Quantum as well as Classical Fidelities for a very special class of Hamiltonians for which the quantum dynamics is exactly solvable in terms of the classical one. For $\epsilon = +1$, the motion is stable and manifests strong periodic recurrences in the classical as well of quantum evolution. It has been exhibited as well in time-periodic systems in [5], and recurrences were already shown for another class of systems in [13].

For $\epsilon = -1$, the classical motion is unstable, although not chaotic. In this case a decrease in time (at $\pm\infty$) occurs for the Quantum as well as the Classical Fidelity functions. However they do not decay to zero, but instead both remain bounded from below. This is important to have here an explicit example where the fidelities (classical as well as quantum) do not decay to zero. This is in contrast with the general "chaotic" situation where the fidelities are generally claimed (although not proven) to decay rapidly to 0.

Qualitatively the behaviour of the Quantum and Classical Fidelities show strong resemblance, except in the neighborhood of $t = 0$ (and of $t = k\pi, \quad k \in \mathbb{Z}$ for the stable case because of periodicity). Namely, instead of being smooth at $t = 0$ the classical fidelities display a cusp. We believe that taking better classical distribution functions in phase-space, and restoring the small parameter $\hbar$ should correct this defect. One might think taking the Wigner functions of the reference wavepackets; however, it is known that only Gaussian wavepackets provide nonnegative Wigner functions which thus mimic a "probability distribution in phase space", and this is not the case here.

Finally we note that, at least in the cases $g = 1, \sqrt{3}, \sqrt{10}$ where we have explicit behaviours of the Quantum Fidelities, these functions remain above the Classical Fidelity functions, in the stable as well as unstable case. However for a better understanding of the correspondence quantum/classical, one has to restore the parameter $\hbar$ and use the semiclassical approach. This is first considered in a rigorous framework (and in a much more general setting) in [6].

8   APPENDIX

Proof of Proposition 4.3
(i) Let us start with the lower bound. Clearly $y'(t)^2 + \dot{y}'(t)^2 \leq q^2 + p^2 + \frac{g^2}{q^2}$, so that

$$F_C(t) \geq \frac{1}{\pi} \int_{\mathbb{R}^2} dq \, dp \, e^{-p^2 - q^2 - \frac{g^2}{q^2}} = e^{-2g} \tag{8.28}$$

uniformly in t, where we use the explicit formula:

$$\int_{\mathbb{R}} dx \, \exp\left(-Ax^2 - \frac{B}{x^2}\right) = \sqrt{\frac{\pi}{A}} e^{-2\sqrt{AB}} \tag{8.29}$$

Of course equ. (4.9) is a much better lower bound, but we have unfortunately no exact computation of the integral. Let us now consider the upper bound.

We denote $q = r \cos \alpha, \; p = r \sin \alpha$, and assume that $r \neq 0$. Then

$$y'(t)^2 + \dot{y}'(t)^2 = r^2 + \frac{2g^2}{r^2 \cos^2 \alpha} - \frac{2g^2}{r^2 \cos^2(t - \alpha) + \frac{2g^2}{r^2 \cos^2 \alpha} \sin^2 t} \tag{8.30}$$

$$\geq r^2 \cos^2(t - \alpha) + \frac{2g^2 \sin^2 t}{r^2 \cos^2 \alpha}$$

This implies:

$$q(t)^2 + p(t)^2 + y'(t)^2 + \dot{y}'(t)^2 \geq p^2 + q^2 + (q \cos t + p \sin t)^2 + \frac{2g^2 \sin^2 t}{q^2} \tag{8.31}$$

Whence

$$F_C(t) \leq \frac{1}{\pi} \int \, dq \, dp \, \exp\left(-\frac{p^2 + q^2}{2} - \frac{1}{2}(q \cos t + p \sin t)^2 - \frac{g^2 \sin^2 t}{q^2} \right) \qquad (8.32)$$

The integral over p can be performed easily:

$$\int \, dp \, \exp\left(-\frac{1}{2}(1 + \sin^2 t)\left[p + \frac{q \cos t \sin t}{1 + \sin^2 t} \right]^2 \right) = \sqrt{\frac{2\pi}{1 + \sin^2 t}} \qquad (8.33)$$

Thus we are left with an integral over q of the form:

$$\int \, dq \, \exp\left(\frac{-q^2}{1 + \sin^2 t} - \frac{g^2 \sin^2 t}{q^2} \right) = \sqrt{\pi(1 + \sin^2 t)} \exp\left(\frac{-g|\sin t|}{\sqrt{1 + \sin^2 t}} \right) \qquad (8.34)$$

This yields the final upper bound which is of course not optimal in the neighborhood of $t = 0 \pmod \pi$.

(ii)

$$\tilde{F}_C(t, g) = \frac{1}{\pi} \int_{\mathbb{R}^2} \, dq \, dp \, \chi(p^2 + q^2 \leq 1) \, \chi(\dot{y}'(t)^2 + y'(t)^2 \leq 1) \qquad (8.35)$$

$$= \frac{1}{\pi} \int \, dq \, dp \, \chi(q^2 + p^2 \leq 1) \, \chi(p^2 + q^2 + \frac{2g^2}{q^2} - \frac{2g^2}{y'^2(t)} \leq 1) \geq \frac{1}{\pi} \int \, dq \, dp \, \chi(p^2 + q^2 + \frac{2g^2}{q^2} \leq 1)$$

$$= \frac{1}{\pi} \int \, dq \, dp \, \chi(p^2 + (q - \frac{g\sqrt{2}}{q})^2 \leq 1 - 2g\sqrt{2})$$

$$= 2 \frac{(g\sqrt{2})^{1/2}}{\pi} \int_{-\infty}^{+\infty} dp \int_0^{+\infty} dq' \, \chi(p^2 + g\sqrt{2}(q' - \frac{1}{q'})^2 \leq 1 - 2g\sqrt{2})$$

$$= \frac{(g\sqrt{2})^{1/2}}{\pi} \int_0^{+\infty} dq' \int_{-\infty}^{+\infty} dp \, (1 + \frac{1}{q'^2}) \, \chi(p^2 + g\sqrt{2}(q' - \frac{1}{q'})^2 \leq 1 - 2g\sqrt{2})$$

$$= \frac{g\sqrt{2}}{\pi} \int_{\mathbb{R}^2} \, dp' \, du \, \chi(p'^2 + u^2 \leq \frac{1 - 2g\sqrt{2}}{g\sqrt{2}}) = 1 - 2g\sqrt{2}$$

where we have used the changes of variables $q = (g\sqrt{2})^{1/2} q'$, $u := q' - \frac{1}{q'}$

(iii) Consider the rescaled variables $q' := (g\sqrt{2})^{-1/2} q$, $p' := (g\sqrt{2})^{-1/2} p$; then

$$\tilde{F}_C(t, g) = \frac{g\sqrt{2}}{\pi} \int_{\mathbb{R}^2} \, dq' \, dp' \, \chi(p'^2 + q'^2 \leq \frac{1}{g\sqrt{2}}) \, \chi(q'^2 + p'^2 + \frac{1}{q'^2} - \frac{1}{y'^2(t)} \leq \frac{1}{g\sqrt{2}})$$
$$(8.36)$$

But

$$\chi(y'^2(t) + \dot{y}'^2(t) \leq \frac{1}{g\sqrt{2}}) \leq \chi(p'^2 + \frac{1}{q'^2} \leq \frac{1}{g\sqrt{2}}) \qquad (8.37)$$

so that

$$\tilde{F}_C(\frac{\pi}{2}, g) \leq \frac{g\sqrt{2}}{\pi} \int_{\mathbb{R}^2} dp' \, dq' \, \chi(p'^2 + q'^2 \leq \frac{1}{g\sqrt{2}}) \, \chi(p'^2 + \frac{1}{q'^2} \leq \frac{1}{g\sqrt{2}}) \tag{8.38}$$

Now, for $g \geq \frac{1}{\sqrt{2}}$ the domains

$$\left\{ (q', p') \in \mathbb{R}^2 : \ p'^2 + q'^2 \leq \frac{1}{g\sqrt{2}} \right\}, \ \text{ and } \ \left\{ (q', p') \in \mathbb{R}^2 : \ p'^2 + \frac{1}{q'^2} \leq \frac{1}{g\sqrt{2}} \right\}$$

have no common domain of positive area, so that

$$\tilde{F}_C(\frac{\pi}{2}, g) = 0 \tag{8.39}$$

□

Proof of Proposition 4.5

Denote:

$$Q(p, q) := p^2 + q^2 + y^2(t) + \dot{y}^2(t) = 2(p^2 + q^2) + \frac{1}{q^2} - \frac{1}{y^2(t)} \tag{8.40}$$

Let us expand $Q(p, q)$ near $t = 0$ up to order 2. We get:

$$Q(p, q) \simeq 2(p^2 + q^2) + 2t\frac{p}{q^3} - \frac{t^2}{2q^2} + \frac{p^2 t^2}{q^4} + \frac{t^2}{q^6} \tag{8.41}$$

Therefore

$$e^{-\frac{Q(p,q)}{2}} \lesssim e^{-(p + \frac{t}{2q^3})^2 - q^2 + \frac{t^2}{4q^2} - \frac{t^2}{4q^6}} \tag{8.42}$$

We thus have:

$$\frac{1}{\pi} \int dp \, dq \, e^{-Q(p,q)/2} \lesssim \frac{1}{\sqrt{\pi}} \int dq \, \exp\left(-q^2 + \frac{t^2}{4q^2} - \frac{t^2}{4q^6} \right) \tag{8.43}$$

Let us divide the integration domain into two parts:
- $q^4 \leq \frac{1}{1+A^2}$
- $q^4 \geq \frac{1}{1+A^2}$

$A > 0$ being an arbitrary constant. Then in the first domain, we have:

$$\frac{1}{4q^2} - \frac{1}{q^6} \leq -\frac{A^2}{4q^2} \tag{8.44}$$

so that the corresponding contribution to equ. (8.43) is approximated by:

$$\frac{1}{\sqrt{\pi}} \int_{q^4 \leq \frac{1}{1+A^2}} dq \, e^{-q^2 - \frac{t^2 A^2}{4q^2}} \tag{8.45}$$

In the second domain we have:

$$-q^2 + \frac{t^2}{4q^2} \le -q^2 \left(1 - \frac{(1+A^2)^2 t^2}{4}\right) - \frac{A^2 t^2}{4q^2} \qquad (8.46)$$

so that the corresponding contribution to equ. (8.43) is

$$\lesssim \frac{1}{\sqrt{\pi}} \int_{q^4 \ge \frac{1}{1+A^2}} dq \, \exp\left(-q^2(1 - \frac{t^2(1+A^2)^2}{4}) - \frac{A^2 t^2}{4q^2}\right) \qquad (8.47)$$

Summing up the two contributions we get as $t \simeq 0$

$$\frac{1}{\pi} \int dp \, dq \, e^{-\frac{Q(p,q)}{2}} \lesssim \frac{1}{\sqrt{\pi}} \int dq \exp\left(-q^2(1 - \frac{t^2(1+A^2)^2}{4}) - \frac{t^2 A^2}{4q^2}\right) \qquad (8.48)$$

$$\simeq \left(1 - \frac{t^2(1+A^2)^2}{4}\right)^{-1/2} \exp\left(-A|t|(1 - \frac{t^2(1+A^2)^2}{4})^{1/2}\right)$$

using the explicit formula (8.29) for any positive constants A, B. Now taking the dominant behavior of the RHS of equ.(75) as $t \simeq 0$ yields the result. $\square$

Proof of Proposition 6.3

Due to conservation of energy for Hamiltonian H_g^-, we have:

$$\dot{y}'^2(t) - y'^2(t) + \frac{2g^2}{y'^2(t)} = p^2 - q^2 + \frac{2g^2}{q^2} \qquad (8.49)$$

so that:

$$\dot{y}'^2(t) + y'^2(t) = 2y'^2(t) + p^2 - q^2 + \frac{2g^2}{q^2} - \frac{2g^2}{y'^2(t)} \qquad (8.50)$$

$$= \cosh 2t(q^2 + p^2 + \frac{2g^2}{q^2}) + 2pq \sinh 2t - \frac{2g^2}{y'^2(t)}$$

$$= \cosh 2t\left((p + q \tanh 2t)^2 + \frac{2g^2}{q^2}\right) + \frac{q^2}{\cosh 2t} - \frac{2g^2}{y'^2(t)}$$

We perform the following changes of variables:

$$v := \sqrt{\cosh 2t}(p + q \tanh 2t), \quad u := \frac{q}{\sqrt{\cosh 2t}} \qquad (8.51)$$

This yields:

$$2y'^2(t) = v^2(1 - \tau) + u^2(1 + \tau) + \frac{2g^2}{u^2}(1 - \tau) + 2uv\sqrt{1 - \tau^2} \qquad (8.52)$$

and

$$\dot{y}'(t)^2 + y'(t)^2 = v^2 + u^2 + \frac{2g^2}{u^2} - \frac{4g^2}{v^2(1 - \tau) + u^2(1 + \tau) + 2uv\sqrt{1 - \tau^2} + (1 - \tau)\frac{2g^2}{u^2}} \qquad (8.53)$$

where we have denoted

$$\tau := \frac{1}{\cosh 2t} \leq 1 \tag{8.54}$$

It clearly follows from (8.53) that

$$\dot{y}'^2(t) + y'^2(t) \leq v^2 + u^2 + \frac{2g^2}{u^2} \tag{8.55}$$

Now in terms of the same variables u, v we have:

$$\dot{x}(t)^2 + x(t)^2 = v^2 + u^2 \tag{8.56}$$

When passing to the new integration variables u, $v \in \mathbb{R}$, we get:

$$G_C(t, g) \geq \frac{1}{\pi} \int_{\mathbb{R}^2} du \, dv \, \exp\left(-u^2 - v^2 - \frac{g^2}{u^2}\right) = e^{-2g} \tag{8.57}$$

using again the formula (8.29).

Now we come to $\tilde{G}_C(t, g)$:

$$\tilde{G}_C(t, g) \geq \frac{1}{3\pi} \int_{\mathbb{R}^2} du \, dv \, \chi\left(u^2 + v^2 + \frac{2g^2}{u^2} \leq 3\right) \tag{8.58}$$

$$= \frac{2}{3\pi} \int_{-\infty}^{+\infty} dv \int_0^{+\infty} du \, \chi\left(v^2 + (u - \frac{g\sqrt{2}}{u})^2 \leq 3 - 2g\sqrt{2}\right)$$

$$= \frac{2(g\sqrt{2})^{1/2}}{3\pi} \int_0^{+\infty} du' \int_{-\infty}^{+\infty} dv \, \chi\left(v^2 + g\sqrt{2}(u' - \frac{1}{u'})^2 \leq 3 - 2g\sqrt{2}\right) \tag{8.59}$$

$$= \frac{(g\sqrt{2})^{1/2}}{3\pi} \int_0^{+\infty} du' \int_{-\infty}^{+\infty} dv \, (1 + \frac{1}{u'^2}) \chi\left(v^2 + g\sqrt{2}(u' - \frac{1}{u'})^2 \leq 3 - 2g\sqrt{2}\right)$$

We now use the new integration variable $x := (g\sqrt{2})^{1/2}(u' - \frac{1}{u'})$ which yields:

$$\frac{1}{3\pi} \int_{\mathbb{R}^2} dx \, dv \, \chi(x^2 + v^2 \leq 3 - 2g\sqrt{2}) = 1 - \frac{2g\sqrt{2}}{3} \tag{8.60}$$

□

Proof of Proposition 6.5

Using the same change of variable $(p, q) \to (u, v)$ as in (8.51), we get:

$$G_C(t) \geq \int_{\mathbb{R}^2} du \, dv \, \exp\left(-u^2 - v^2 - \frac{1}{2u^2}\right) = e^{-\sqrt{2}} \tag{8.61}$$

and

$$\tilde{G}_C(t) \geq \frac{1}{3\pi} \int_{\mathbb{R}^2} du\, dv\, \chi\left(u^2 + v^2 + \frac{1}{u^2} \leq 3\right) = \frac{1}{3\pi} \int_{\mathbb{R}^2} du\, dv\, \chi\left(v^2 + (u - \frac{1}{u})^2 \leq 1\right) \tag{8.62}$$

$$= \frac{1}{3\pi} \int_0^{+\infty} du \int_{-\infty}^{+\infty} dv\, (1 + \frac{1}{u^2})\, \chi\left(v^2 + (u - \frac{1}{u})^2 \leq 1\right)$$

$$= \frac{1}{3\pi} \int_{\mathbb{R}^2} dv\, dx\, \chi(x^2 + v^2 \leq 1) = \frac{1}{3}$$

□

(ii) The quantity $\dot{y}(t)^2 + y(t)^2$ (with the scaled variables as in Proposition 6.1) can be rewritten as in (8.53) using the variables u, v, τ defined by (8.51) and (8.54):

$$\dot{y}(t)^2 + y(t)^2 = u^2 + v^2 + \frac{1}{u^2} - \frac{2}{v^2(1 - \tau) + u^2(1 + \tau) + (1 - \tau)\frac{1}{u^2} + 2uv\sqrt{1 - \tau^2}} \tag{8.63}$$

so that as $t \to \infty$, $\tau \to 0$ very rapidly and we get

$$\tilde{G}_C(\infty) = \frac{1}{3\pi} \int_{\mathbb{R}^2} du\, dv\, \chi(u^2 + v^2 \leq 3)\, \chi\left(u^2 + v^2 + \frac{1}{u^2} - \frac{2}{(u + v)^2 + \frac{1}{u^2}} \leq 3\right) \tag{8.64}$$

But since

$$\left\{(u,\, v):\ u^2 + v^2 + \frac{1}{u^2} - \frac{2}{(u + v)^2 + \frac{1}{u^2}}\right\} \subset \left\{(u,\, v):\ u^2 + v^2 + \frac{1}{u^2} - 2u^2 \leq 3\right\} \tag{8.65}$$

$$\tilde{G}_C(\infty) \leq \frac{1}{3\pi} \int_{\mathbb{R}^2} du\, dv\, \chi(u^2 + v^2 \leq 3)\, \chi\left(v^2 - u^2 + \frac{1}{u^2} \leq 3\right) \tag{8.66}$$

$$\leq \frac{1}{3\pi} \int_{\mathbb{R}^2} du\, dv\, \chi(u^2 + v^2 \leq 3)\, \chi\left(\frac{1}{u^2} \leq 6\right)$$

This means that the estimate equals the part of the disk in u, v space of radius $\sqrt{3}$ outside of the interval $|q| \leq \frac{1}{\sqrt{6}}$, divided by 3π. This equals precisely

$$\frac{6\arctan(\sqrt{17}) - \sqrt{17}/3}{3\pi} = \frac{2\arctan(\sqrt{17})}{\pi} - \frac{\sqrt{17}}{9\pi} \tag{8.67}$$

□

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